Int. J. Solids Structures, 1968, Vol. 4, pp. 1221 to 1231. Pergamon Press. Printed in Great Britain

TRANSIENT RESONANCE OF AN IDEAL STRING UNDER A LOAD MOVING WITH VARYING SPEED

F. T. FLAHERTY, JR.

Bell Telephone Laboratories, Inc.
Whippany, New Jersey

Abstract—The transverse velocity of an ideal string is obtained when the string experiences a point load moving with varying speed. Two cases are evaluated in which the load either accelerates or decelerates through the characteristic string speed. The consequent momentary resonance is examined and the results show that as the load speed passes through the string speed a singularity in transverse string velocity is produced which then propagates at the string speed as a homogeneous wave. The string displacements remain continuous.

INTRODUCTION

IT is well known that certain continuous elastic systems exhibit a kind of resonance when they experience loads, moving at some constant characteristic speed, for example, an ideal string under a transverse load moving at the characteristic string speed [1], or a line load moving over the surface of an elastic half-space at the Rayleigh speed [2, 3]. One of the important properties of this resonance phenomenon has been investigated, that of the resonant buildup observed after the load traveling at constant characteristic speed is first applied [1, 3]. However, for a resonance of this type, another property is also of importance and apparently has not been investigated, that is, the momentary resonant coupling existing at the instant a load traveling at continuously varying speed passes through the characteristic speed of the body. Such a momentary resonance probably occurs as an explosively produced blast wave propagates over the surface of the earth, and in dissipating, slows down through the Rayleigh speed of the soil.

In the present paper this second resonant property is examined for a simple situation. In particular, the transverse velocity of an infinite, ideal string is obtained for two cases: in Case I, a transverse point load appears at the origin at zero time and then moves along the string with speed decreasing hyperbolically with distance through the characteristic string speed; in Case II, a transverse point load appears at the origin at zero time and then accelerates parabolically through the string speed. In both cases, a singularity in transverse string velocity is generated the instant the load traverses the string speed, the singularity thereafter propagating at the string speed as a homogeneous wave. These singularities are integrable and the string displacements remain continuous. In the Appendix, the solution for each case is examined in the limit as the speed of the load becomes constant and the expected results are attained.

The method used is essentially a simplified form of Ang's [4] version of Cagniard's method [5].

GENERAL DEVELOPMENT

Given a uniform, infinite, taut string of tension T, assume that a transverse point load $\mathscr{F}$ appears at the origin of coordinates at $t = 0$ and then moves to the right with some varying speed $U(x)$ (Fig. 1).

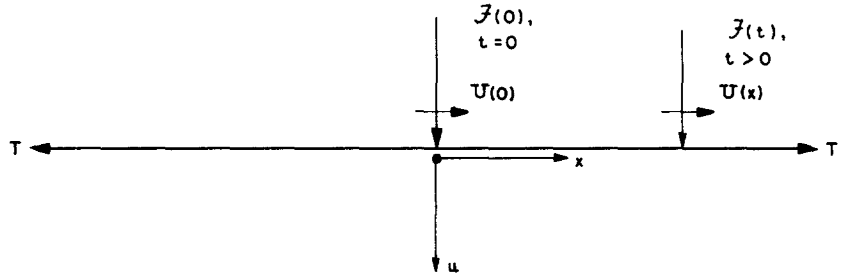

FIG. 1. Schematic of the string problem.

The well-known ideal dynamic string equation may be written

$$\begin{aligned} \ddot{u} &= a^2 u_{xx} + \frac{F}{\rho}\,\delta[t - f(x)], \qquad x \geq 0 \\ \ddot{u} &= a^2 u_{xx}, \qquad x < 0 \end{aligned} \tag{1}$$

where u is the transverse string displacement, $a^2 = T/\rho$, ρ = lineal density of the string, F = amplitude of the point load, δ = Dirac delta function and the subscript x denotes a partial derivative with respect to x and dots with respect to t. $f(x)$ is the load arrival time and is defined by

$$f(x) = \int_0^x \frac{\mathrm{d}\eta}{U(\eta)}. \tag{2}$$

The resultant load $\mathscr{F}$ of Fig. 1 is not a constant in time by virtue of the form of the force density assumed in (1). Instead

$$\mathscr{F}(t) = F \int_{-\infty}^{\infty} \delta[t - f(x)]\,\mathrm{d}x = F/|f'(x_0)|$$

where the prime denotes differentiation with respect to the argument and x_0 is defined by $f(x_0) = t$.

Taking the Laplace transform of (1) with respect to time, assuming zero initial conditions, gives

$$\begin{aligned} s^2 \bar{u} &= a^2 \bar{u}_{xx} + \frac{F}{\rho}\,\mathrm{e}^{-sf(x)}, \qquad x \geq 0 \\ s^2 \bar{u} &= a^2 \bar{u}_{xx}, \qquad x < 0 \end{aligned} \tag{3}$$

where in general

$$\bar{G}(s) = \int_0^{\infty} G(t)\,\mathrm{e}^{-st}\,\mathrm{d}t. \tag{4}$$

Then applying Fourier transformation with respect to x, i.e.

$$G^*(\omega) = \int_{-\infty}^{\infty} G(x)\,\mathrm{e}^{-i\omega x}\,\mathrm{d}x, \tag{5}$$

(3) may be written

$$\frac{\rho \bar{u}^*}{F} = \frac{1}{s^2 + a^2\omega^2}\int_0^{\infty} \mathrm{e}^{-[sf(\xi) + i\omega\xi]}\,\mathrm{d}\xi. \tag{6}$$

Using the inverse Fourier transform corresponding to (5), and changing the order of integration, (6) becomes

$$\frac{\rho \bar{u}}{F} = \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-[sf(\xi) + i\omega\xi - i\omega x]}}{s^2 + a^2\omega^2}\,\mathrm{d}\omega\,\mathrm{d}\xi. \tag{7}$$

By making the substitution $p = i\omega/s$ (7) becomes

$$\frac{\rho \bar{u}}{F} = \frac{-1}{2\pi i}\int_0^{\infty}\int_{-i\infty}^{i\infty} \frac{\mathrm{e}^{-s[f(\xi) + p(\xi - x)]}}{a^2 s(p - 1/a)(p + 1/a)}\,\mathrm{d}p\,\mathrm{d}\xi. \tag{8}$$

Since only forward Laplace transformations are to be considered, s can be assumed real and positive. Then performing the simple contour integration over p in (8), keeping the sign of $\xi - x$ in mind, yields

$$\begin{aligned}\frac{2a\rho s\bar{u}}{F} &= \int_0^{x} \mathrm{e}^{-s[f(\xi) - (1/a)(\xi - x)]}\,\mathrm{d}\xi \\ &\quad + \int_x^{\infty} \mathrm{e}^{-s[f(\xi) + (1/a)(\xi - x)]}\,\mathrm{d}\xi, \qquad x > 0\end{aligned} \tag{9}$$

$$\frac{2a\rho s\bar{u}}{F} = \int_0^{\infty} \mathrm{e}^{-s[f(\xi) + (1/a)(\xi - x)]}\,\mathrm{d}\xi \qquad x < 0. \tag{10}$$

Substituting t for the quantities in brackets in the exponents of (9, 10) gives, respectively,

$$\frac{2a\rho s\bar{u}}{F} = \int_{x/a}^{f(x)} \dot{\xi}_1(t)\,\mathrm{e}^{-st}\,\mathrm{d}t + \int_{f(x)}^{\infty} \dot{\xi}_2(t)\,\mathrm{e}^{-st}\,\mathrm{d}t, \qquad x > 0 \tag{11}$$

$$\frac{2a\rho s\bar{u}}{F} = \int_{-x/a}^{\infty} \dot{\xi}_2(t)\,\mathrm{e}^{-st}\,\mathrm{d}t, \qquad x < 0, \tag{12}$$

where ξ_1 and ξ_2 are given by the relations

$$t = f(\xi_1) - (\xi_1 - x)/a$$

and (13)

$$t = f(\xi_2) + (\xi_2 - x)/a.$$

The lower limit takes on this simple form since $f(0) = 0$.

The inverse Laplace transformation of (11, 12) may then be obtained by inspection since the equations are written in the form of the forward Laplace transformation (4) and since the zero initial conditions assumed here allow $s\bar{u} = \bar{\dot{u}}$. Thus, for $x > 0$

$$\frac{2a\rho\dot{u}}{F} = \begin{cases} \dot{\xi}_1(t), & x/a < t < f(x) \\ \dot{\xi}_2(t), & t > f(x) \\ 0, & \text{on other intervals} \end{cases} \tag{14}$$

and for $x < 0$

$$\frac{2a\rho\dot{u}}{F} = \begin{cases} \dot{\xi}_2(t), & t > -x/a \\ 0, & \text{on other intervals} \end{cases} \tag{15}$$

and the solution is obtained in general except at a finite number of isolated points.

The integration limits in (11, 12) and the interval limits in (14, 15) can only be considered as starting and end points for t. It will be shown, for example, that t may first decrease from the lower limit along one branch of the inverse function and then increase to the upper limit along another branch. This condition arises because, in evaluating ξ_1 and ξ_2 from (13) explicitly, the proper branches must be used to ensure that ξ traverses the intervals defined by the limits in (9, 10) as t travels from one limit to the other in (11, 12).

Case I, $U = c/x$

From (2)

$$f(x) = \int_0^x \frac{d\eta}{U(\eta)} = \frac{x^2}{2c},$$

so that from (13),

$$\begin{aligned} \xi_1 &= \frac{c}{a} \pm \left[\frac{c^2}{a^2} + 2c\left(t - \frac{x}{a}\right)\right]^{\frac{1}{2}}, \\ \xi_2 &= -\frac{c}{a} \pm \left[\frac{c^2}{a^2} + 2c\left(t + \frac{x}{a}\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{16}$$

The proper branches for ξ_1 are as follows: If $x < c/a$, the branch with the minus sign applies, and t decreases from x/a to $x^2/2c$. If $x > c/a$, the branch with the minus sign applies as t decreases from x/a to the branch point $x/a - c/2a^2$; then the branch with the plus sign is appropriate as t increases from $x/a - c/2a^2$ to $x^2/2c$. The proper branch for ξ_2 in (11) is that with the plus sign over the entire range so that (14) becomes, for $x > 0$,

$$\frac{2a\rho\dot{u}}{F} = A' + A'' + B'$$

where

$$A' = \begin{cases} A, & x < c/a \quad \text{and} \quad x^2/2c < t < x/a \\ & \text{or } x > c/a \quad \text{and} \quad x/a - c/2a^2 < t < x/a \\ 0, & \text{on other intervals} \end{cases} \tag{17}$$

$$A'' = \begin{cases} A, & x > c/a \quad \text{and} \quad x/a - c/2a^2 < t < x^2/2c \\ 0, & \text{on other intervals} \end{cases} \tag{18}$$

$$B' = \begin{cases} B, & t > x^2/2c \\ 0, & \text{on other intervals} \end{cases} \tag{19}$$

$$A = c\left[\frac{c^2}{a^2} + 2c\left(t - \frac{x}{a}\right)\right]^{-\frac{1}{2}},$$

and

$$B = c\left[\frac{c^2}{a^2} + 2c\left(t + \frac{x}{a}\right)\right]^{-\frac{1}{2}}.$$

The proper branch for ξ_2 in (12) is also that with the plus sign over the entire range so that (15) becomes for $x < 0$,

$$\frac{2a\rho\dot{u}}{F} = \begin{cases} B, & t > -x/a \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Normalized time-of-arrival plots for the homogeneous wave fronts and the load are presented in Fig. 2. Also indicated are the regions of influence of the contributing solutions,

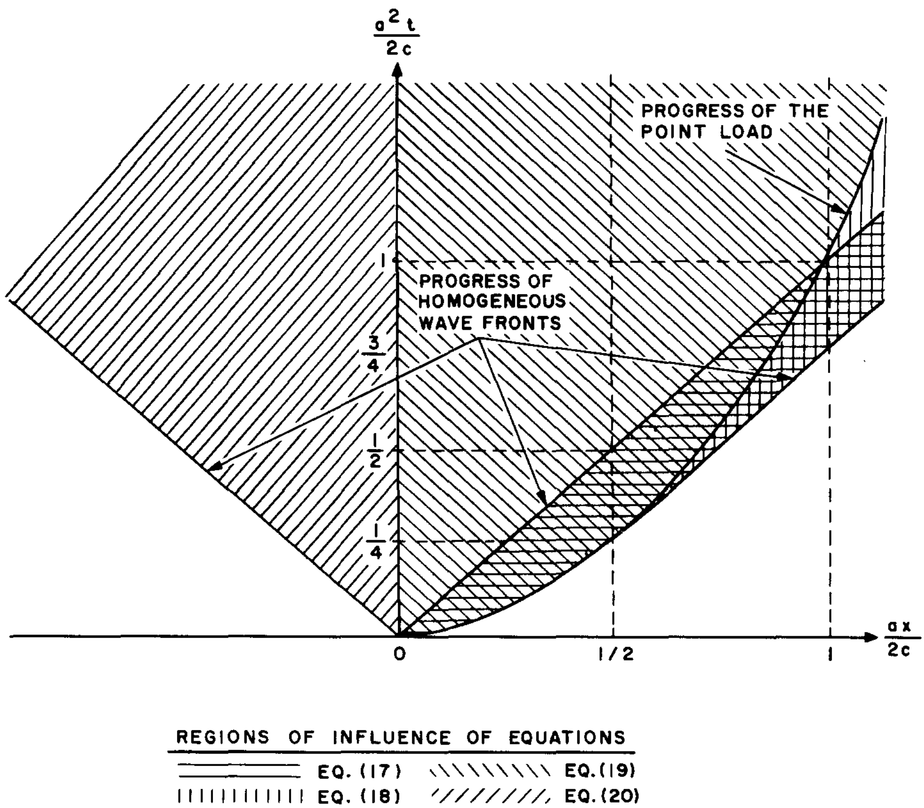

FIG. 2. Case I, time-of-arrival diagram ($a^2t/2c$) vs. ($ax/2c$).

(17–20). The normalized velocity distributions of the string are plotted in Fig. 3 for four instants of time. We note that the resonance occurs at $a^2t/2c = 0{\cdot}25$ and $ax/2c = 0{\cdot}5$ corresponding to $U = a$.

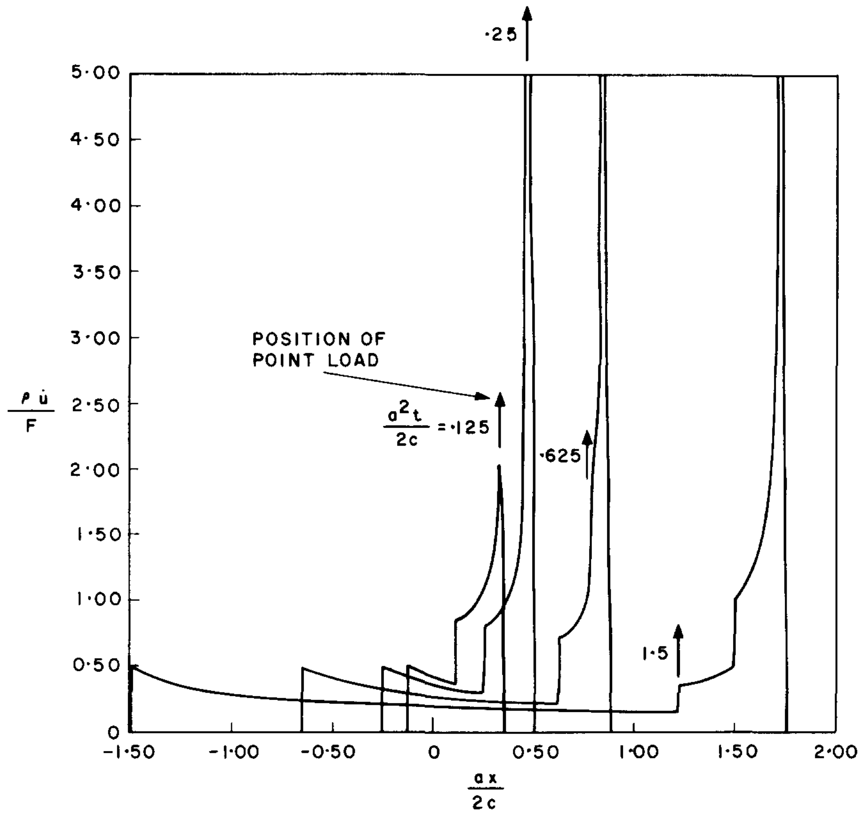

FIG. 3. Case I, normalized velocity ($\rho\dot{u}/F$) vs. distance ($ax/2c$) for four values of time ($a^2t/2c$).

It is shown in the Appendix that the present solution reduces to the expected form as $c \to 0$. An interesting feature of this special case is that it is qualitatively like the string problem with the loading $F\delta(t)\delta(x)$ but with only one δ-function so that the resulting velocity spikes have finite amplitude, the string momentum thus remains zero, and no displacements are generated.

Case II, $U = (cx+b)^2$

The extra parameter b must be introduced so that the initial speed of the point load is nonzero. However, in order that the point load speed increases through the string speed, we require that $\sqrt{a} > b > 0$. For U to be monotonically increasing with x, we also take $c > 0$. Then from (2)

$$f(x) = \int_0^x \frac{d\eta}{(c\eta+b)^2} = \frac{x/b}{cx+b}. \tag{21}$$

From (13)

$$\xi_1 = \frac{1}{2}\left[x - at + \frac{a}{bc} - \frac{b}{c}\right] \pm \left[\frac{1}{4}\left(x - at + \frac{a}{bc} - \frac{b}{c}\right)^2 + \frac{b}{c}(x - at)\right]^{\frac{1}{2}},$$

and

$$\xi_2 = \frac{1}{2}\left[x + at - \frac{a}{bc} - \frac{b}{c}\right] \pm \left[\left(x + at - \frac{a}{bc} - \frac{b}{c}\right)^2 + \frac{b}{c}(at + x)\right]^{\frac{1}{2}}. \tag{22}$$

The proper branches for ξ_1 are as follows: If $x < (\sqrt{a} - b)/c$, the branch with the minus sign applies as t increases from x/a to $x/b(cx+b)$. If $x > (\sqrt{a} - b)/c$, t increases from x/a to $(x/a + b/ca + 1/bc - 2/c\sqrt{a})$ along the branch with the minus sign and then decreases from this latter value to $x/b(cx+b)$ along the branch with the plus sign. The proper branch for ξ_2 in (11) is that with the plus sign over the entire interval. Then (14) becomes, for $x > 0$

$$\frac{2a\rho\dot{u}}{F} = C' + C'' + D'$$

where

$$C' = \begin{cases} -a/2 + C, x < (\sqrt{a} - b)/c & \text{and} \quad x/a < t < x/b(cx+b) \\ \quad\quad \text{or } x > (\sqrt{a} - b)/c & \text{and} \quad x/a < t < x/a + b/ca + 1/bc - 2/c\sqrt{a} \\ 0, \quad \text{on other intervals} \end{cases} \tag{23}$$

$$C'' = \begin{cases} a/2 + C, & x > (\sqrt{a} - b)/c \quad \text{and} \\ & x/b(cx+b) < t < x/a + b/ca + 1/bc - 2/c\sqrt{a} \\ 0, & \text{on other intervals} \end{cases} \tag{24}$$

$$D' = \begin{cases} a/2 + D, & t > x/b(cx+b) \\ 0, & \text{on other intervals} \end{cases} \tag{25}$$

where

$$C = \frac{a}{2}\left(x - at + \frac{a}{bc} + \frac{b}{c}\right)\left[\left(x - at + \frac{a}{bc} - \frac{b}{c}\right)^2 + 4\frac{b}{c}(x - at)\right]^{-\frac{1}{2}}$$

and

$$D = \frac{a}{2}\left(x + at - \frac{a}{bc} + \frac{b}{c}\right)\left[\left(x + at - \frac{a}{bc} - \frac{b}{c}\right)^2 + 4\frac{b}{c}(x + at)\right]^{-\frac{1}{2}}.$$

The branch for ξ_2 in (15) is again that with the plus sign over the entire interval so that (15) becomes, for $x < 0$,

$$\frac{2a\rho\dot{u}}{F} = \begin{cases} a/2+D, & t > -x/a \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Normalized time-of-arrival plots and regions of influence of the equations for this case are shown in Fig. 4 assuming that $a/b^2 = 4$. The normalized velocity distributions of the string are plotted in Fig. 5 for four instants of time and $a/b^2 = 4$. In this case the resonance

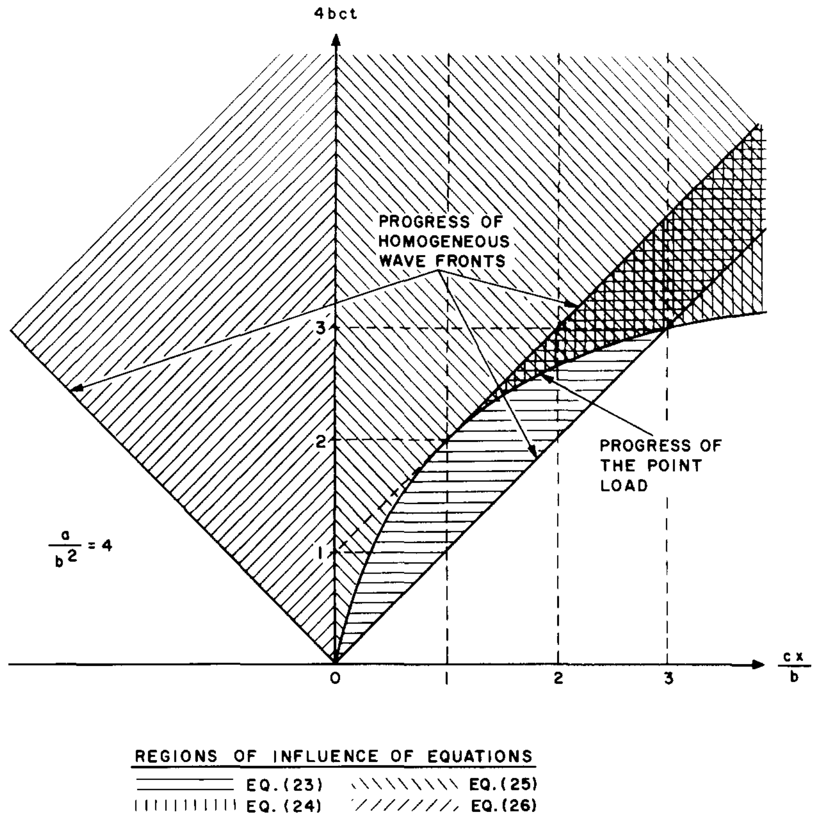

FIG. 4. Case II, time-of-arrival diagram ($4bct$) vs. (cx/b).

occurs at $4bct = 2$ and $ax/ac = 0{\cdot}5$ again corresponding to $U = a$. It is shown in the Appendix that the solution for this case also takes on the expected form when $c = 0$, that is, when the load propagates with constant subsonic speed b^2.

CONCLUSIONS

The problem of a string undergoing excitation by a moving load of nonuniform speed has been solved for two cases in which the speed of the point load either accelerates or decelerates through the characteristic string speed.

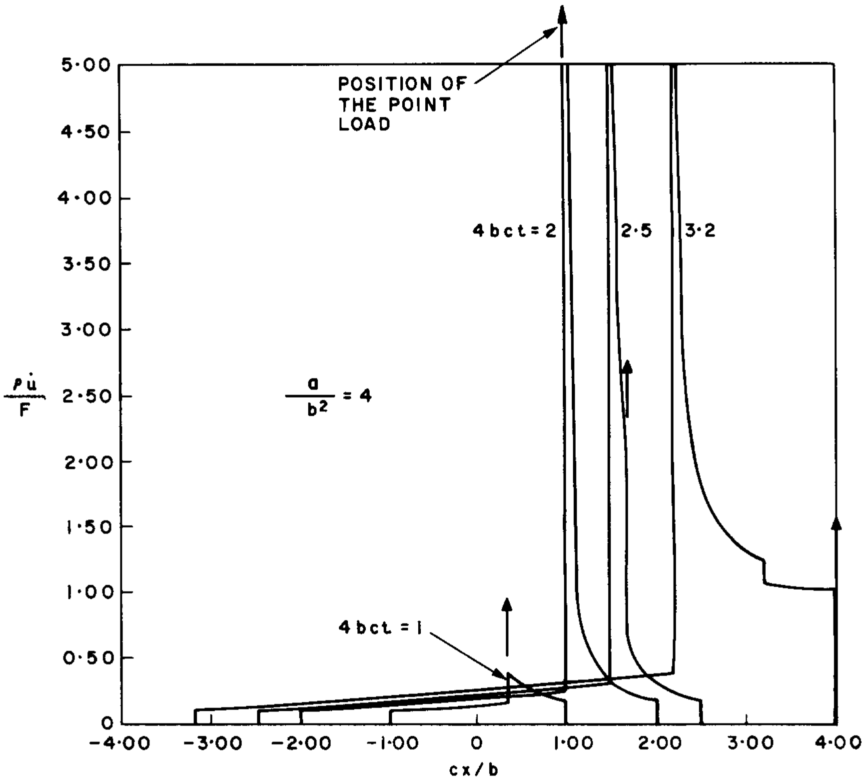

FIG. 5. Case II, normalized velocity ($\rho\dot{u}/F$) vs. distance (cx/b) for four values of time ($4bct$).

Examination of (17, 18, 23, 24), Fig. 3 or Fig. 5 shows that as the speed of the load traverses the characteristic string speed a singularity in transverse string velocity is thrown off which then propagates as a homogeneous wave at the string speed. Since the integrals of $\dot{u}$ over t through this singularity exist in the conventional sense, the displacements remain continuous with only a vertical slope in u at the singular point of $\dot{u}$ to indicate, in the displacements, the existence of this phenomenon.

REFERENCES

[1] M. M. KANNINEN and A. L. FLORENCE, Traveling forces on strings and membranes. *Int. J. Solids Struct.* **3**, 143–154 (1967).

[2] J. W. CRAGGS, On two-dimensional waves in an elastic half-space. *Proc. Camb. phil. Soc. math. phys. Sci.* **56**, 269–285 (1960).

[3] R. V. GOL'DSHTEIN, Rayleigh waves and resonance phenomena in elastic bodies. *J. appl. Math. Mech.* **29**, 608–619 (1965).

[4] D. D. ANG, Transient motion of a line load on the surface of an elastic half-space. *Q. appl. Math.* **XVIII**, 251–256 (1960).

[5] L. CAGNIARD, *Reflection and Refraction of Progressive Seismic Waves*. McGraw-Hill (1962).

[6] A. N. TIKHONOV and A. A. SAMARSKII, *Equations of Mathematical Physics*. Pergamon Press (1963).

APPENDIX

Examination of the results for the two cases in the limit as the constant $c \to 0$ shows that these results attain the expected forms.

Case I, $U = c/x$

The form of the point load for this case is $F\delta[t-x^2/2c]$. In the limit as $c \to 0$, the load becomes

$$F\delta[t-\infty] = 0, \qquad x \neq 0 \tag{A1}$$

$$F\delta[t], \qquad x = 0, \tag{A2}$$

and thus it represents a Dirac δ function in time acting at the origin only. This can be treated as an initial value problem in which an infinitesimal length of string at the origin has an initial velocity corresponding to the impulse applied by the point load. This initial velocity is clearly

$$\frac{\text{impulse}}{\text{mass}} = \lim_{\Delta x \to 0} \frac{\int_{-\infty}^{\infty} F\delta[t]\, dt\, \Delta x}{\rho \Delta x} = \frac{F}{\rho}. \tag{A3}$$

The solution to this problem is well known (e.g. [6] p. 45) and consists of two velocity pulses of infinitesimal width and amplitude $F/2\rho$ propagating, one in each direction, away from the origin at the string speed a. Examination of all the equations contributing to the solution for this case (17–20) shows that they are all zero everywhere except for (17) which gives, when $c = 0$

$$\dot{u} = \begin{cases} F/2\rho, & x > 0, t = x/a \\ 0, & \text{otherwise}, \end{cases} \tag{A4}$$

and (20) which gives, when $c = 0$

$$\dot{u} = \begin{cases} F/2\rho, & x < 0, t = -x/a \\ 0, & \text{otherwise}. \end{cases} \tag{A5}$$

These are the expected results.

It is interesting to note here that the usual forcing function used for an impulsive point load at the origin is $F\delta(t)\delta(x)$. The well-known solution for the velocity for this loading is

$$\dot{u} = \begin{cases} \dfrac{F}{2a\rho}\delta(t+ax), & x < 0 \\ \dfrac{F}{2a\rho}\delta(t-ax), & x > 0 \end{cases}$$

and the solution for the displacement is a step wave of amplitude $F/2a\rho$ propagating away from the origin in both directions at the string speed a. The solution for the present problem with $c = 0$ has the same qualitative form but since the forcing function has only one δ-function, the velocity pulses are finite, the impulse and the momentum are zero, and the displacement remains zero everywhere.

Case II, $U = (cx+b)^2$, $\sqrt{a} > b > 0$, $c > 0$

In this case, the forcing function becomes, for $c = 0$,

$$F\delta\left[t - \frac{x}{b^2}\right]. \tag{A6}$$

(A6) represents a point load which appears at the origin at $t = 0$ and propagates towards increasing x with the constant subsonic speed b^2. Although a simple problem, for comparison, its solution will be presented here. Inserting $f(x) = x/b^2$ into (1) and taking the Laplace transform as defined in (4), (1) becomes

$$\begin{aligned} s^2\bar{u} &= a^2\bar{u}_{xx}+\frac{F}{\rho}\,\mathrm{e}^{-xs/b^2}, \qquad x \geq 0 \\ s^2\bar{u} &= a^2\bar{u}_{xx}, \qquad x < 0. \end{aligned} \tag{A7}$$

(A7) may be rewritten in terms of the Laplace transform of the transverse string speed $s\bar{u}$ as

$$\begin{aligned} s(s\bar{u}) &= \frac{a^2}{s}(s\bar{u})_{xx}+\frac{F}{\rho}\,\mathrm{e}^{-xs/b^2}, \qquad x \geq 0 \\ s(s\bar{u}) &= \frac{a^2}{s}(s\bar{u})_{xx}, \qquad x < 0. \end{aligned} \tag{A8}$$

The Laplace inversion of the solution to (A8) which is appropriate for large $\pm x$ in the two regions and which provides continuity for $s\bar{u}$ and $(s\bar{u})_x$ at $x = 0$ is

$$\begin{aligned} \frac{2\rho\dot{u}}{F} &= \frac{H(t-x/a)}{(a/b^2)-1}-\frac{2H(t-x/b^2)}{(a^2/b^4)-1}, \qquad x > 0 \\ \frac{2\rho\dot{u}}{F} &= \frac{H(t+x/a)}{(a/b^2)+1}, \qquad x < 0. \end{aligned} \tag{A9}$$

where H is the Heaviside step function.

Examination of all the contributions to this case (23–26) shows that when $c = 0$, (24) yields no contribution, (23, 25) together reduce to the first of (A9), and (26) reduces to the second of (A9). Thus again, the present solutions reduce to the expected results.

(*Received* 19 *February* 1968)

Абстракт—Получается поперечная скорость идеальной струны, когда она подвергается сосредоточенной нагрузке, движущейся с переменной скоростью. Выводятся два случая, в которых нагрузки либо ускоряется, либо уменьшается в сравнении с характеристической скоростью струны. Исследуется последовательный моментной резонанс. Результаты указывают, что когда скорость нагрузки переходит через скорость струны, тогда вызывается сингулярность в поперечной скорости струны, которая затем распространяется, при скорости струны, как однородная волны. Перемещения струны остаются непрерывны.